2.2 Semantics of Predicate Logic

Montag, 13. April 2015 08:30

- Ex. Sheet 1 on the web (due on Apr. 20)
- FR, April 17: 2 lectures (lecture instead of exercise course)
- MO, April 20: ex. course instead of lecture
- register for exercises on our website (until Fri, April 17)

2.2 Semantics of Pred. Logic

Goal: Describe the meaning of formulas and terms

Use interpretations I:

function

assigns a meaning α_f to every fct. symbol f

and — " — α_p — " — pred. symbol p

relation

Def 2.2.1 (Interpretation, Structure, Satisfiability, Model)

Let (Σ, Δ) be a signature. An interpretation for (Σ, Δ) is a triple $I = (\mathcal{A}, \alpha, \beta)$. $\mathcal{A}$ is the carrier of the interpr. where $\mathcal{A}$ is a set with $\mathcal{A} \neq \emptyset$. The mapping α maps every $f \in \Sigma_n$ to a function $\alpha_f : \mathcal{A}^n \to \mathcal{A}$ and every $p \in \Delta_n$ with $n > 0$ to a set $\alpha_p \subseteq \mathcal{A}^n$. For $p \in \Delta_0$, we have $\alpha_p \in \{TRUE, FALSE\}$. Here, α_f and α_p are the meaning of f and p, resp. The mapping $\beta : \mathcal{V} \to \mathcal{A}$ is called a variable assignment.

For every interpretation I, we get a function

$I : \mathcal{T}(\Sigma, \mathcal{V}) \to \mathcal{A}$:

$I(X) = \beta(X)$ for all $X \in \mathcal{V}$

$I(f(t_1, \dots, t_n)) = \alpha_f(I(t_1), \dots, I(t_n))$

Ex 2.2.2 Consider the following interpretation:

$I = (\mathcal{A}, \alpha, \beta)$ with

$\mathcal{A} = \mathbb{N}$

$\alpha_n = n$ for all $n \in \mathbb{N}$

$\alpha_{monika} = 0, \alpha_{karin} = 1, \alpha_{renate} = 2, \dots$

$\alpha_{date}(n_1, n_2, n_3) = n_1 + n_2 + n_3$ for all $n_1, n_2, n_3 \in \mathbb{N}$

$\alpha_{female} = \{ n \mid n \text{ is even} \}, \alpha_{male} = \{ n \mid n \text{ is odd} \},$

$\alpha_{human} = \mathbb{N}, \alpha_{married} = \{ (n, m) \mid n > m \}, \dots$

$\beta(X) = 0, \beta(Y) = 1, \beta(Z) = 2, \dots$

Meaning of the term $date(1, X, karin)$ under this interpr:

$I(date(1, X, karin)) = \alpha_{date}(\alpha_1, \beta(X), \alpha_{karin})$

$= 1 + 0 + 1 = 2$

Def 2.2.1 (contd.)

For $X \in \mathcal{V}$ and $a \in \mathcal{A}$, let $\beta[\![X/a]\!]$ be the var. assignment with $\beta[\![X/a]\!](X) = a$

$\beta[\![X/a]\!](Y) = \beta(Y)$ for all $Y \neq X$

Similarly, for $I = (\mathcal{A}, \alpha, \beta)$, let

$I[\![X/a]\!] = (\mathcal{A}, \alpha, \beta[\![X/a]\!])$.

An interpretation $I = (\mathcal{A}, \alpha, \beta)$ satisfies a formula $\varphi \in \mathcal{F}(\Sigma, \Delta, \mathcal{V})$ (denoted $I \models \varphi$) iff

- $\varphi = p(t_1, \dots, t_n)$ with $p \in \Delta_n$ with $n > 0$ and $(I(t_1), \dots, I(t_n)) \in \alpha_p$ or
- $\varphi = p$ with $p \in \Delta_0$ and $\alpha_p = TRUE$ or
- $\varphi = \neg \varphi_1$ and $I \not\models \varphi_1$ or

- $\varphi = \varphi_1 \wedge \varphi_2$ and $I \models \varphi_1$ and $I \models \varphi_2$ or
- $\varphi = \varphi_1 \vee \varphi_2$ and ($I \models \varphi_1$ or $I \models \varphi_2$) or
- $\varphi = \varphi_1 \rightarrow \varphi_2$ and if $I \models \varphi_1$, then $I \models \varphi_2$ or
- $\varphi = \varphi_1 \leftrightarrow \varphi_2$ and ($I \models \varphi_1$ iff $I \models \varphi_2$) or
- $\varphi = \forall X\ \varphi_1$ and $I[\![X/a]\!] \models \varphi_1$ for all $a \in \mathcal{A}$ or
- $\varphi = \exists\ \varphi_1$ and $I[\![X/a]\!] \models \varphi_1$ for some $a \in \mathcal{A}$

Ex 222 (contd.)

$I \models \text{marriedOf}(\text{date}(1, X, \text{Karin}), \text{Karin})$

iff $(\underbrace{I(\text{date}(1,X,\text{Karin}))}_{2}, \underbrace{I(\text{Karin})}_{\alpha_{\text{Karin}}=1}) \in \alpha_{\text{marriedOf}}$

$\alpha_{\text{marriedOf}} = \{(n,m) \mid n > m\}$

$\Rightarrow I$ satisfies this formula, since $2 > 1$.

$I \models \forall X\ \text{female}(\text{date}(X,X,\text{monika}))$.

iff $I[\![X/a]\!] \models \text{female}(\text{date}(X,X,\text{monika}))$ for all $a \in \mathcal{A}$

$\mathcal{A} = \mathbb{N}$

iff $I[\![X/a]\!](\text{date}(X,X,\text{monika})) \in \alpha_{\text{female}}$ for all $a \in \mathbb{N}$

iff $\alpha_{\text{date}}(a, a, \alpha_{\text{monika}}) \in \alpha_{\text{female}}$ for all $a \in \mathbb{N}$

$\alpha_{\text{monika}} = 0$, $\alpha_{\text{female}} = \{n \mid n \text{ is even}\}$

iff $a + a + 0 \in \{n \mid n \text{ is even}\}$ for all $a \in \mathbb{N}$

true!

Def 221 (contd.)

An interpret. I is a model of φ iff $I \models \varphi$. I is a model of $\Phi \subseteq \mathcal{F}(\Sigma, \Delta, \mathcal{V})$

iff $\mathcal{I} \models \varphi$ for all $\varphi \in \Phi$. (We write $\mathcal{I} \models \Phi$)

Two formulas φ_1, φ_2 are equivalent iff $\mathcal{I} \models \varphi_1$ iff $\mathcal{I} \models \varphi_2$ for all interpretations $\mathcal{I}$

A formula (or set of formulas) is satisfiable iff it has a model. It is called valid iff it is satisfied by every interpretation.

An interpretation $S = (\mathcal{A}, \alpha)$ without var. assignment is called structure. For closed formulas, it suffices to regard structures:

$S \models \varphi$ iff $\mathcal{I} \models \varphi$ for some interpretation $\mathcal{I} = (\mathcal{A}, \alpha, \beta)$

Similarly, we can define $S(t)$ for ground terms t.

The formulas in Ex. 2.22 were satisfiable, but not valid (there exist interpretations that don't satisfy them).

Example for a valid formula:

$\varphi \vee \neg\varphi$ for any $\varphi \in \mathcal{F}(\Sigma, \Delta, \mathcal{V})$

Ex. for unsatisfiable formula:

$\varphi \wedge \neg\varphi$

Lemma 2.2.3 clarifies the connections between:

substitution $\sigma : \mathcal{V} \to \mathcal{T}(\Sigma, \mathcal{V})$ – syntax

Variable assignment $\beta : \mathcal{V} \to \mathcal{A}$ – semantics

Lemma 2.2.3 (Substitution Lemma)

Let $\mathcal{I} = (\mathcal{A}, \alpha, \beta)$ be an interpretation, let $\sigma = \{X_1/t_1, \dots, X_n/t_n\}$ be a substitution.

(a) $I(\sigma(t)) = I[\![X_1/I(t_1),\dots,X_n/I(t_n)]\!](t)$

(b) $I \models \sigma(\varphi)$ iff

$I[\![X_1/I(t_1),\dots,X_n/I(t_n)]\!] \models \varphi$.

Ex 224 Let I be the interpr. from Ex. 222.

Let $\sigma = \{X/date(1,X,karin)\}$, let $t = date(X,Y,Z)$.

$I(\sigma(t)) = I(date(date(1,X,karin),Y,Z))$

$= \alpha_{date}(\alpha_{date}(\underset{1}{\alpha_1},\underset{0}{\beta(X)},\underset{1}{\alpha_{karin}}),\underset{1}{\beta(Y)},\underset{2}{\beta(Z)})$

$= 1+0+1+1+2$

$= 5$

$I[\![X/\underbrace{I(date(1,X,karin))}_{\alpha_{date}(\alpha_1,\beta(X),\alpha_{karin})=2}]\!](date(X,Y,Z)) =$

$I[\![X/2]\!](date(X,Y,Z)) =$

$\alpha_{date}(\underbrace{\beta[\![X/2]\!](X)}_{2},\underbrace{\beta[\![X/2]\!](Y)}_{1},\underbrace{\beta[\![..]\!](Z)}_{2})$

$= 5$

Proof of the substitution lemma 223:

(a) To prove: for all terms t we have

$I(\sigma(t)) = I[\![X_1/I(t_1),\dots,X_n/I(t_n)]\!](t)$

To prove such statements on inductively defined data structures (like terms): structural induction

(As ind. hypothesis, one may assume that the statement already holds for direct subterms.)

Ind. Base: $t \in \mathcal{V}$

Case 1: $t \in \{X_1,\dots,X_n\}$, e.g. $t = X_i$

$I(\sigma(X_i)) = I(t_i)$

$I[\![X_1/I(t_1),\dots,X_n/I(t_n)]\!](X_i) = I(t_i)$ ✓

Case 2 $t \in \mathcal{V} \setminus \{X_1, \dots, X_n\}$, e.g. $t = Y$

$I(\sigma(Y)) = I(Y)$

$I[\![X_1/I(t_1), \dots]\!](Y) = I(Y)$ ✓

Ind. Step: $t = f(s_1, \dots, s_k)$ (where $k=0$ is possible)

$I(\sigma(f(s_1, \dots, s_k))) =$

$I(f(\sigma(s_1), \dots, \sigma(s_k))) =$

$\alpha_f(\underline{I(\sigma(s_1))}, \dots, \underline{I(\sigma(s_k))})$

$I[\![X_1/I(t_1), \dots]\!](f(s_1, \dots, s_k)) =$

$\alpha_f(\underline{I[\![X_1/I(t_1), \dots]\!](s_1)}, \dots, \underline{I[\![X_1/I(t_1), \dots]\!](s_k)})$

Ind. Hypothesis:
$I(\sigma(s_i)) = I[\![X_1/I(t_1), \dots]\!](s_i)$ for all $i \in \{1, \dots, k\}$ ✓

(b) analogous to (a)

$\boxed{\not{\!\!\!\!\mathcal{E}}}$

Def 225 (Entailment)
A set of formulas Φ entails the formula φ (denoted $\Phi \models \varphi$) iff

$I \models \Phi$ implies $I \models \varphi$ for all interpretations I.

Instead of "$\emptyset \models \varphi$" we also write "$\models \varphi$".

means: φ is valid

Ex 226 Entailment is checked when executing logic programs: $\Phi \models \varphi$

program clauses query

If Φ are the clauses for the example program, then ?- male(gerd).

means that we want to check

$$\Phi \models male(gerd) \qquad \text{holds} \checkmark$$

The query ?- human(gerd)

means $\Phi \models human(gerd)$.

This holds, because:

$$I \models \Phi$$

$$\rightsquigarrow \quad I \models \forall X \; human(X)$$

$$\rightsquigarrow \quad I[\![X/a]\!] \models human(X) \quad \text{for all } a \in A$$

$$\rightsquigarrow \quad I[\![X/I(gerd)]\!] \models human(X)$$

$$\rightsquigarrow \quad I \models human(gerd) \quad \text{by the subst. lemma.}$$